

A Corrected Value-at-Risk Predictor*

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Abstract

In this note it is argued that the estimation error in Value-at-Risk predictors gives rise to underestimation of portfolio risk. We propose a simple correction and find in an empirical illustration that it is economically relevant.

Key Words: Estimation Error, Finance, GARCH, Prediction, Risk Management.

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1 Introduction

Value-at-Risk (VaR) has become a standard measure of market risk and it is widely used by financial institutions and their regulators. The VaR is defined as the maximum potential portfolio loss that will not be exceeded with a given probability, or

$$\Pr \{ \text{portfolio loss} \geq VaR^{1-\alpha} \} = \alpha. \quad (1)$$

The Basel Committee on Banking Supervision, which includes governors of the main central banks, imposes on financial institutions such as banks and investment firms to meet capital requirements based on VaR . Accurate VaR estimates are therefore crucially important, and VaR has already received much attention in the literature (see Jorion (2007) for a survey). Although the literature dealing with different modeling issues is large, surprisingly little is written about the uncertainty of VaR predictors. Three studies attempting to quantify the uncertainty are Jorion (1996), Christoffersen and Gonçalves (2005) and Chan, Deng, Peng, and Xia (2007).

The question a practitioner naturally poses is how uncertainty in the VaR affects risk management, i.e. does it in some way change what value to report. Tsay (2005, ch. 7) points out that VaR should be computed using the predictive distribution of returns and should take into account the parameter uncertainty in a properly specified model. The uncertainty arises from two primary sources. The true data generating process is not known, which gives rise to model risk, and the parameters of the hypothesized model must be estimated, which gives rise to estimation risk.

The focus of this paper is on how to incorporate the estimation error in the VaR predictor. In particular, we take a time series model and demonstrate that the implied conventional plug-in VaR predictor does not satisfy eq. (1) asymptotically. In fact, if $VaR^{1-\alpha}$ in eq. (1) is replaced by its predictor, a stochastic variable, the corresponding probability is higher than α , i.e. the portfolio risk is underestimated. This is of course an undesirable feature, but it is relatively straightforward to correct the predictor to give the correct risk measure interpretation. We propose a corrected VaR predictor that accounts for estimation risk. Schaller (2002) discusses along similar lines and suggests an alternative approach. We emphasize that the correction is due to the randomness of the VaR predictor and it is not due to conventional bias, i.e. that the expected value of the VaR predictor is different from the true

value. Two studies attempting to correct for conventional bias are Bao and Ullah (2004) and Hartz, Mitnik, and Paolella (2006).

2 VaR and uncertainty

A general multivariate time series model with conditional mean and variance is

$$\mathbf{y}_t = \boldsymbol{\mu}(\boldsymbol{\theta}_1, \Psi_{t-1}) + \mathbf{H}^{1/2}(\boldsymbol{\theta}_2, \Psi_{t-1}) \boldsymbol{\varepsilon}_t, \quad t = 1, \dots, T, \quad (2)$$

where Ψ_{t-1} is the information set at $t - 1$, $\boldsymbol{\varepsilon}_t \sim i.i.d.(\mathbf{0}, \mathbf{I})$ and $\boldsymbol{\mu}$ and \mathbf{H} are, respectively, vector and matrix valued functions. In the sequel we occasionally use the shorthand notation $\boldsymbol{\mu}_t = \boldsymbol{\mu}(\boldsymbol{\theta}_1, \Psi_{t-1})$ and $\mathbf{H}_t = \mathbf{H}(\boldsymbol{\theta}_2, \Psi_{t-1})$.

Models of this type are usually estimated by maximizing the log-likelihood function $\ln L_T(\boldsymbol{\theta}) \propto -\frac{1}{2} \sum_t^T [\ln |\mathbf{H}_t| + (\mathbf{y}_t - \boldsymbol{\mu}_t)' \mathbf{H}_t^{-1} (\mathbf{y}_t - \boldsymbol{\mu}_t)]$, i.e. assuming $\boldsymbol{\varepsilon}_t \sim n.i.d.(\mathbf{0}, \mathbf{I})$. Given some regularity conditions the estimator vector, $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_1', \hat{\boldsymbol{\theta}}_2')'$, is asymptotically, normally distributed with the true parameter vector, $\boldsymbol{\theta}_0$, as its mean and covariance matrix $\boldsymbol{\Sigma} = -[E(\partial^2 \ln L_T(\boldsymbol{\theta}_0) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}')]^{-1}$.

For a portfolio of assets with returns generated by model (2), the one-period-ahead conditional VaR , $VaR_{T+1}^{1-\alpha}$, satisfies $\Pr(\mathbf{w}_T' \mathbf{y}_{T+1} \leq -VaR_{T+1}^{1-\alpha} | \Psi_T) = \alpha$, where \mathbf{w}_T is a vector of portfolio weights that remains unchanged between T and $T + 1$. Assuming normally distributed errors a predictor of the VaR is

$$\widehat{VaR}_{T+1}^{1-\alpha} = -\mathbf{w}_T' \boldsymbol{\mu}(\hat{\boldsymbol{\theta}}_1, \Psi_T) - \Phi^{-1}(\alpha) \sqrt{\mathbf{w}_T' \mathbf{H}(\hat{\boldsymbol{\theta}}_2, \Psi_T) \mathbf{w}_T}. \quad (3)$$

Since the parameters of the underlying model are estimated, $\widehat{VaR}_{T+1}^{1-\alpha}$ is subject to estimation error and is therefore random. It can be decomposed as

$$\widehat{VaR}_{T+1}^{1-\alpha} = VaR_{T+1}^{1-\alpha} + e_{T+1} + b_{T+1},$$

where e_{T+1} accounts for sampling variation and has zero mean and variance $\sigma_{VaR, T+1}^2$. The finite sample bias of the predictor is denoted b_{T+1} . In a related study, Hansen (2006) showed that asymptotically $\sqrt{T}e_{T+1} \xrightarrow{d} N(0, T\sigma_{VaR, T+1}^2)$, where $\sigma_{VaR, T+1}^2 =$

$(\partial VaR_{T+1}^{1-\alpha}/\partial\theta')\Sigma(\partial VaR_{T+1}^{1-\alpha}/\partial\theta)$. Due to e_{T+1} and b_{T+1} , $\widehat{VaR}_{T+1}^{1-\alpha}$ satisfies

$$\Pr\left\{\mathbf{w}'_T\mathbf{y}_{T+1} \leq -\widehat{VaR}_{T+1}^{1-\alpha}|\Psi_T\right\} = \alpha^*,$$

where α^* need not equal α .

Now, introduce a correction term c_{T+1} such that $\Pr\{\mathbf{w}'_T\mathbf{y}_{T+1} \leq -(VaR_{T+1}^{1-\alpha} + e_{T+1} + b_{T+1} + c_{T+1})|\Psi_T\} = \alpha$. Assume that \mathbf{y}_{T+1} and e_{T+1} are independent¹ and we have that

$$\Pr\left\{\frac{\mathbf{w}'_T\mathbf{y}_{T+1} + e_{T+1} - \mathbf{w}'_T\boldsymbol{\mu}_{T+1}}{\sqrt{\mathbf{w}'_T\mathbf{H}_{T+1}\mathbf{w}_T + \sigma_{VaR,T+1}^2}} \leq \frac{-(VaR_{T+1}^{1-\alpha} + b_{T+1} + c_{T+1}) - \mathbf{w}'_T\boldsymbol{\mu}_{T+1}}{\sqrt{\mathbf{w}'_T\mathbf{H}_{T+1}\mathbf{w}_T + \sigma_{VaR,T+1}^2}}|\Psi_T\right\} = \alpha.$$

By the normality of e_{T+1} and \mathbf{y}_{T+1} the correction can be obtained as

$$c_{T+1} = -\Phi^{-1}(\alpha) \left[\sqrt{\mathbf{w}'_T\mathbf{H}_{T+1}\mathbf{w}_T + \sigma_{VaR,T+1}^2} - \sqrt{\mathbf{w}'_T\mathbf{H}_{T+1}\mathbf{w}_T} \right] - b_{T+1}.$$

Due to b_{T+1} , the correction may in small samples be either positive or negative. Asymptotically, however, b_{T+1} is zero, c_{T+1} is positive, and then $\alpha^* > \alpha$. Now, add the estimator of the correction to the conventional predictor, $\widehat{VaR}_{T+1}^{1-\alpha}$, and the corrected VaR predictor becomes

$$\widehat{CVaR}_{T+1}^{1-\alpha} = -\mathbf{w}'_T\hat{\boldsymbol{\mu}}_{T+1} - \Phi^{-1}(\alpha) \sqrt{\mathbf{w}'_T\hat{\mathbf{H}}_{T+1}^{1/2}\mathbf{w}_T + \hat{\sigma}_{VaR,T+1}^2}.$$

3 A numerical illustration

To illustrate the properties of the correction we conduct a small simulation experiment with data generated according to a GARCH(1,1) model: $y_t = \sqrt{h_t}\varepsilon_t$, with $h_t = 40/252 + 0.1y_{t-1}^2 + 0.8h_{t-1}$ and $\varepsilon_t \sim n.i.d.(0,1)$. We consider samples of sizes 250, 500, 1000, and 2500 observations, $\alpha = 0.01$ and results are in each case based on 100 000 replications. The fractions of exceedences for the conventional (α^*) and the corrected VaR predictor (α^c) are computed and the hypotheses $\alpha^* = \alpha$ and $\alpha^c = \alpha$ are tested against the one-sided alternatives $\alpha^* > \alpha$ and $\alpha^c > \alpha$, respectively. Table

¹In the simulation exercise in Section 3, the covariance between the \mathbf{y}_{T+1} and the e_{T+1} series was close to zero for all four sample sizes.

Table 1: Statistics for the corrections in the numerical illustration. S.D. is the standard deviation. The z 's are the test statistics for the hypotheses $\alpha^* = 0.01$ and $\alpha^c = 0.01$, respectively.

T	Mean	S.D.	Skewness	Kurtosis	z_{α^*}	z_{α^c}
250	0.061	0.063	5.007	60.236	3.909	-0.318
500	0.028	0.027	4.272	34.852	1.716	-0.222
1000	0.013	0.013	4.196	31.151	0.254	-0.636
2500	0.005	0.005	4.304	34.284	0.032	-0.254

1 gives some statistics for the estimated corrections and the z -statistics for the tests.

As expected the mean and the variance of the correction decreases with the sample size. The hypothesis $\alpha^* = \alpha$ is rejected at the 5% level for the sample sizes 250 and 500, while the hypothesis $\alpha^c = \alpha$ is not rejected for any of the four sample sizes. That is, the correction matters statistically for the two smaller sample sizes and it appears that it does the job of taking the estimation error into account.

4 An empirical illustration

VaR corrections are obtained for the three major stock market indices: FTSE 100 of UK, Nikkei 225 of Japan, and S&P 100 of USA. Five years of daily index data were downloaded from DataStream and returns were calculated as $y_t = 100 \times \log(I_t/I_{t-1})$, where I_t is the value of the index at t . The sample covers February 6, 2003 to February 7, 2008, for a total of 1304 observations. We consider $\alpha = 0.01$ and the predictor for VaR at $t + 1$ is based on observations $t - 653$ to t , $t = 654, \dots, 1304$. VaR 's are predicted for the final half of the sample, and are based on re-estimated GARCH(1, 1) models with constant means. Figure 1 gives the estimated corrections in percentage points.

The corrections exhibit time variation and vary between 0.003 and 0.140 for the FTSE 100 index, 0.003 and 0.069 for Nikkei 225 and 0.002 and 0.183 for S&P 100. The average corrections are 0.016, 0.014 and 0.015. These small numbers must be converted into monetary units to give a fair picture. For example, a correction of 0.05 for a portfolio with 100 billion dollars worth of assets is 50 million dollars on a daily basis.

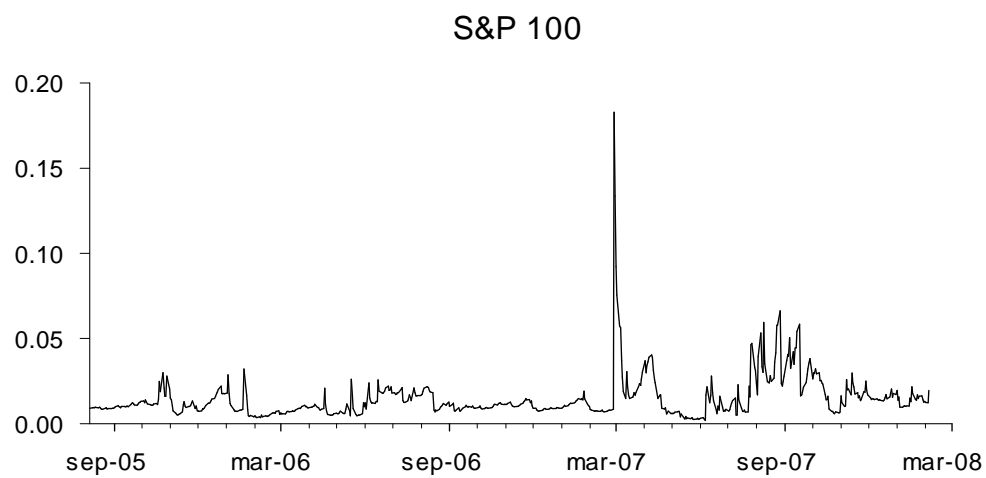
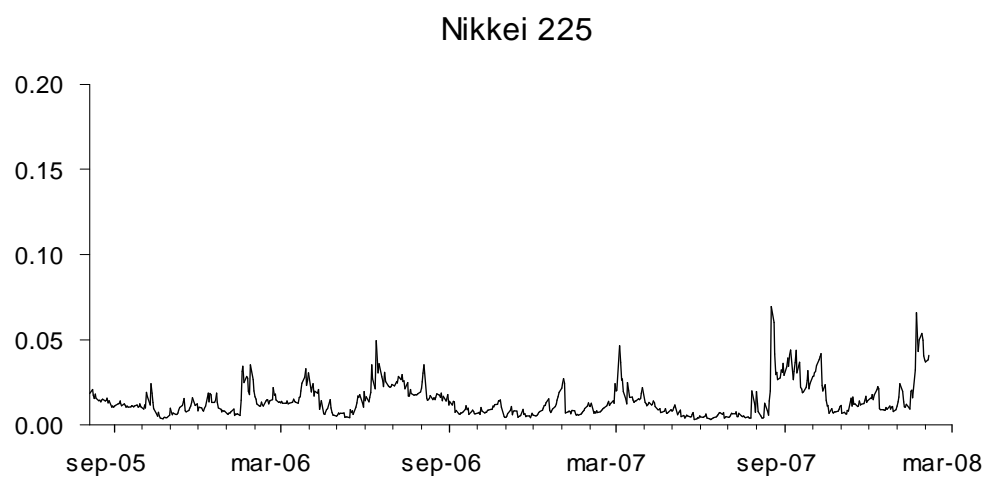
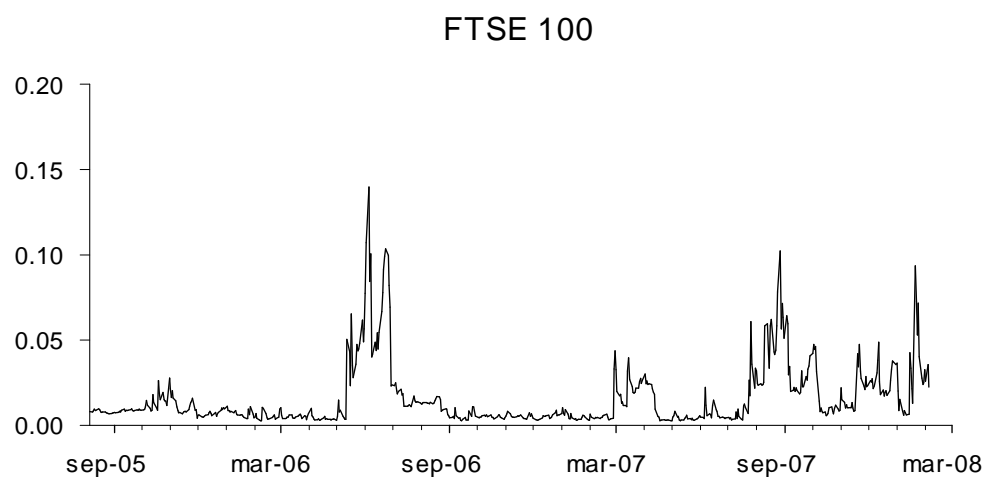


Figure 1: Corrections in percentage points.

The few relatively large corrections are due to outliers and highlight the sensitivity of both the Quasi-Maximum Likelihood and the $\sigma_{VaR,T+1}^2$ estimators to extreme observations.

5 Conclusion

This note argued that the estimation error in *VaR* predictors gives rise to underestimation of portfolio risk. We introduced an approach to correcting a predictor to account for the estimation error, and in an empirical illustration we found that the correction is of economic relevance. The proposed correction hinges on the normality of both the *VaR* estimator and the returns and does not apply directly to cases with non-normally distributed returns. Adapting the proposed approach to other distributions is in principle straightforward, though.

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